From One Finite Operation to Univalent Moduli: Circle Topology, Orbit–Stabilizer Groupoids, and Rezk Completion

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A single binary operation on three states can support an unexpectedly long ascent through modern mathematics. Starting from an anonymous 3×3 table, the construction first generates the full positive-arity clone, then exposes a two-element meet algebra through explicit derived terms. A selected free construction yields a seven-object self-indexing world, whose thin-category nerve contains a deleted boundary equivalent to the circle. From that circle follow an explicit universal cover, winding number, fundamental group, cyclic covers, monodromy, integral local systems, and representations of the integers. In a second branch, the six relabelings of the seed form an action groupoid under S_3. The seed is rigid, so its stabilizer is trivial, and the orbit–stabilizer equivalence reduces the moduli groupoid to a terminal skeleton. This terminal category is Rezk-complete and satisfies the bounded identity–isomorphism correspondence characteristic of univalent categories. The result does not derive full homotopy type theory from one operation. It gives a finite, exact, and reproducible passage from universal algebra to topology, groupoids, homotopy quotients, and a 1-truncated univalent moduli completion. Every transition separates derived theorem, explicit witness, selected policy, and classical consequence.

Keywords: finite universal algebra, functionally complete clone, three-element binary operation, theory-preserving ontogenesis, derived meet-semilattice, self-indexing mathematical world, thin category, simplicial nerve, circle topology, universal cover, fundamental groupoid, cyclic covers, monodromy, integral local systems, representation theory, symmetric group S_3, group actions, orbit–stabilizer groupoid, classifying space, homotopy quotient, moduli groupoid, rigid algebra, Rezk completeness, univalent category, homotopy type theory, computational mathematics, reproducible proof.

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Code: https://github.com/DougYouvan/theory-preserving-ontogenesis-v2.0.1-seed-univalent-completion

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1. From the Finite Seed to the Seven-Object World

1.1 Primitive Data

Let

$A = \{0,1,2\}$

and let $w: A \times A \to A$ be the binary operation

w = 201/122/010.

Equivalently,

w(0,0)=2, w(0,1)=0, w(0,2)=1,

w(1,0)=1, w(1,1)=2, w(1,2)=2,

w(2,0)=0, w(2,1)=1, w(2,2)=0.

The symbols 0, 1, and 2 are labels. No arithmetic, order, topology, identity element, or algebraic law is assumed. The initial language contains only variables, projections, and composition through w.

Terms are generated recursively by

x_i

and

w(s,t),

where s and t are previously formed terms. Two terms are identified when they induce the same function on A.

The initial datum is therefore one finite carrier and one binary table.

1.2 Full Expressive Closure

Let Clo(w) denote the positive-arity clone generated by w.

The unary closure contains all

$3^3 = 27$

functions $A \to A$.

The binary closure contains all

$3^9 = 19,683$

functions $A^2 \to A$.

The cumulative unary closure counts are

$1 \to 2 \to 5 \to 21 \to 27$.

The cumulative binary closure counts are

$2 \to 6 \to 38 \to 1{,}256 \to 19{,}681 \to 19{,}683$.

The seed is surjective and depends essentially on both inputs. Since its clone contains every unary function, the finite functional-completeness criterion applies. Hence

$\mathrm{Clo}(w) = O_a$,

where O_a is the full clone of all positive-arity operations on A.

Therefore, for every $n \geq 1$ and every function

$f: A^n \to A$,

there exists a term t constructed only from w and variables such that

$t^A = f$.

The seed is not merely expressive enough to imitate selected algebraic structures. It generates every finitary operation of positive arity on its carrier.

1.3 An Explicit Retraction

Functional completeness guarantees the existence of many internal structures, but it does not select one. The next step requires explicit terms.

Define

$r(x) = w(w(x,x),w(x,x))$.

Direct evaluation gives

$r(0)=0$,

$r(1)=0$,

$r(2)=2$.

Thus

$r^2 = r$

and

B = r(A) = {0,2}.

The map r is an idempotent retraction from A onto a two-element image. The state 1 is identified with 0 by an operation generated internally from the original seed.

No quotient relation or two-state algebra is imposed externally. The reduction is witnessed by a unary term in the language of w.

1.4 A Derived Meet Operation

Define

m(x,y) = w(w(x,x),w(y,x)).

Restricted to B, the table is

m(0,0)=0,

m(0,2)=0,

m(2,0)=0,

m(2,2)=2.

Therefore, on B,

m(x,x)=x,

m(x,y)=m(y,x),

m(m(x,y),z)=m(x,m(y,z)).

The element 0 is absorbing and 2 is an identity. Hence

(B,m) ≅ 2,

where 2 denotes the two-element meet-semilattice.

The induced order is

0 ≤ 2,

with

x ≤ y ⇔ m(x,y)=x.

The seed has therefore produced an internal two-state order algebra through explicit term formation.

The bridge may be summarized as

$(A,w) \vdash r,m,$

$r(A)=B,$

$(B,m) \cong (\{0,2\},\wedge).$

The symbol $\vdash$ denotes term definability. No new primitive operation has been added.

1.5 The Selected Free World

The next transition is a construction policy rather than a theorem forced uniquely by w.

Define

$\Omega := F^{+}\wedge(A),$

the free nonempty meet-semilattice on the three elements of A.

Its elements are the nonempty finite meets of three generators. Therefore,

$|\Omega| = 2^3 - 1 = 7.$

Using subset notation,

$\Omega \cong P(A) - \{\emptyset\}.$

The seven objects are

$\{0\},$

$\{1\},$

$\{2\},$

$\{0,1\},$

$\{0,2\},$

$\{1,2\},$

$\{0,1,2\}.$

The meet operation is union,

$S \wedge T = S \cup T,$

and the order is reverse inclusion,

$S \leq T \Leftrightarrow S \supseteq T.$

The singleton subsets are the three free generators. The full set A is the least element.

The same structure has a coordinate representation in the derived algebra B. For every nonempty $S \subseteq A$, define

$\chi_s(a)=0$ if $a \in S$,

$\chi_s(a)=2$ if $a \notin S$.

Then

$\Omega \cong B^A - \{\top^A\}$,

where $\top^A$ is the constant function with value 2.

Thus Ω is the three-dimensional binary cube with its constant-top vertex removed.

1.6 Self-Indexing Reconstruction

For each $a \in A$, define a coordinate homomorphism

$v_a: \Omega \to B$

by

$v_a(S)=0$ if $a \in S$,

$v_a(S)=2$ if $a \notin S$.

Each v_a preserves meet because

$v_a(S \cup T)=v_a(S) \wedge v_a(T)$.

Collect the three views into

$V: \Omega \to B^A$,

$V(S)=(v_0(S),v_1(S),v_2(S))$.

The map V is injective. Its image is exactly

$B^A - \{\top^A\}$.

Hence the seven-object world is reconstructed by three binary observations.

Two binary views cannot separate seven objects because they provide at most four response patterns. Three binary views provide eight patterns, seven of which occur. Therefore, the minimum binary reconstruction dimension of Ω is

3.

The construction is self-indexing because the original carrier A returns in three distinct roles:

A as the carrier of w,

A as the free generator set of Ω,

A as the index set of the separating coordinate views.

The initial states become the indices by which the later world is reconstructed.

1.7 Logical Status of the Construction

The developmental chain contains three distinct kinds of transition.

First,

$w \vdash Clo(w)=O_a$

is a theorem about the seed.

Second,

$w \vdash r,m$

gives explicit internal witnesses.

Third,

$\Omega := F^{+}\Lambda(A)$

is a declared construction choice.

These claims must remain distinct. Functional completeness does not uniquely select the retraction, the meet algebra, or the free world. The chosen bridge is short and explicit. The chosen world is canonical relative to the stated free-semilattice policy.

The foundational chain is therefore

one binary operation

→ full positive-arity clone

→ explicit two-state retract

→ derived meet algebra

→ selected seven-object free world

→ exact reconstruction by three binary views.

This is the point reached by the earlier development. The present paper begins its new mathematics by treating Ω not only as a finite algebra, but also as a category whose nerve carries topology.

2. Categorifying the Self-Indexing World

2.1 The Poset as a Thin Category

The seven-object world Ω is a meet-semilattice ordered by reverse inclusion:

$S \leq T \Leftrightarrow S \supseteq T.$

Every poset determines a thin category. The objects are the elements of the poset, and there is a unique morphism

$S \to T$

exactly when

$S \leq T.$

Thus, in Ω,

$S \to T \Leftrightarrow S \supseteq T.$

There is at most one morphism between any ordered pair of objects. Identity morphisms follow from reflexivity, and composition follows from transitivity.

This categorical interpretation adds no new primitive data. It changes the level at which the structure is read. The same seven objects now support composable arrows, chains, simplices, and a nerve.

The full-support object

$A = \{0,1,2\}$

is least in the reverse-inclusion order. Therefore it has a unique arrow to every object of Ω.

The singleton objects

$\{0\}, \{1\}, \{2\}$

are maximal. No nonidentity arrow leaves a singleton.

The two-element subsets occupy the intermediate level. For example,

$\{0,1,2\} \to \{0,1\} \to \{0\}$

and

$\{0,1,2\} \to \{0,1\} \to \{1\}$.

The category therefore has three ranks:

one bottom object,

three pair objects,

three singleton objects.

Its Hasse diagram is the incidence structure between the vertices and edges of a triangle, together with one global cone point beneath the entire diagram.

2.2 The Simplicial Nerve

Let $N(\Omega)$ denote the nerve of the thin category Ω.

An n-simplex of $N(\Omega)$ is a chain of n composable morphisms:

$S_0 \to S_1 \to \cdots \to S_n$.

Equivalently,

$S_0 \supseteq S_1 \supseteq \cdots \supseteq S_n$.

Nondegenerate simplices correspond to strict chains:

$S_0 \supsetneq S_1 \supsetneq \cdots \supsetneq S_n$.

Because Ω has three ranks, its nondegenerate simplices occur only in dimensions 0, 1, and 2.

The nondegenerate 0-simplices are the seven objects.

The nondegenerate 1-simplices are the strict comparable pairs. There are twelve:

six arrows from A to the two-element subsets and singletons,

and six arrows from two-element subsets to their contained singletons.

More explicitly, A has arrows to all six proper nonempty subsets. Each two-element subset has arrows to its two singleton subsets.

The nondegenerate 2-simplices are the strict chains

$A \to \{i,j\} \to \{i\}$

and

$A \to \{i,j\} \to \{j\}$.

There are six such chains.

Hence the nondegenerate f-vector is

(7,12,6).

No nondegenerate simplex has dimension greater than two.

2.3 Face Maps

For a nondegenerate 2-simplex

$A \rightarrow \{i,j\} \rightarrow \{i\}$,

the three faces are obtained by deleting one object from the chain:

$d_0(A,\{i,j\},\{i\}) = (\{i,j\},\{i\})$,

$d_1(A,\{i,j\},\{i\}) = (A,\{i\})$,

$d_2(A,\{i,j\},\{i\}) = (A,\{i,j\})$.

Thus each triangular simplex joins:

the cone point A,

one pair object,

one singleton object.

The six 2-simplices fill the six sectors between the bottom object and the six-object boundary.

The simplicial boundary of an oriented 2-simplex is

$\partial[S_0,S_1,S_2]$

$[S_1,S_2] - [S_0,S_2] + [S_0,S_1]$.

The simplicial identities hold because deletion of entries in a chain satisfies

$d_i d_j = d_{j-1} d_i$

whenever $i < j$.

The nerve therefore converts categorical composition into simplicial incidence.

2.4 The Cone Structure

The object A maps uniquely to every object of Ω. It is an initial object of the thin category.

A category with an initial object has contractible nerve. In the present finite case, this can be seen directly.

Every simplex

$S_0 \to S_1 \to \cdots \to S_n$

can be extended on the left by A whenever A is not already present:

$A \to S_0 \to S_1 \to \cdots \to S_n$.

Geometrically, every simplex lies in a simplex joined to the vertex A. Therefore $|N(\Omega)|$ is a cone with apex A.

Hence

$|N(\Omega)| \simeq *$.

The full seven-object nerve is contractible.

This conclusion is important. The topology does not appear by merely taking the nerve of Ω. The complete world contains a universal lower object, and that object collapses the realization to a point.

The nontrivial topology is exposed only after a declared deletion.

2.5 Removing the Cone Point

Define

$\Omega\partial = \Omega - \{A\}$.

The objects of Ωð are the six proper nonempty subsets of A:

{0}, {1}, {2},

{0,1}, {0,2}, {1,2}.

The induced order remains reverse inclusion.

The only nonidentity morphisms are

{0,1} → {0},

{0,1} → {1},

{0,2} → {0},

{0,2} → {2},

{1,2} → {1},

{1,2} → {2}.

There are no composable pairs of nonidentity arrows. The reduced category therefore has no nondegenerate 2-simplices.

Its nerve has:

six vertices,

six edges,

no higher nondegenerate simplices.

The resulting graph alternates between pair objects and singleton objects:

{0}
←
{0,1}
→
{1}
←
{1,2}
→
{2}
←
{0,2}
→
{0}.

Ignoring arrow orientation in the geometric realization, this is a cycle of length six.

Thus

$N(\Omega\partial) \cong C_6$

as a one-dimensional simplicial complex.

2.6 The Boundary of the Triangle

The six-object category Ωð has a standard geometric interpretation.

The singleton subsets

{0}, {1}, {2}

represent the three vertices of a triangle.

The two-element subsets

{0,1}, {0,2}, {1,2}

represent its three edges.

A morphism from a pair object to a singleton object records incidence of an edge with one of its endpoints.

The nerve of this incidence poset is the barycentric subdivision of the boundary of the 2-simplex:

$N(\Omega\partial) \cong Sd(\partial\Delta^2)$.

Barycentric subdivision preserves geometric realization. Therefore

$|N(\Omega\partial)| \cong |\partial\Delta^2|$.

Since the boundary of the 2-simplex is a circle,

$|N(\Omega\partial)| \cong S^1$.

The circle is not inserted as an external geometric object. It is recovered from the incidence relations already present among the six proper objects of the self-indexing world.

2.7 Homology of the Deleted Boundary

Let C_0 and C_1 be the simplicial chain groups of $N(\Omega\partial)$ with integer coefficients.

Since there are six vertices and six edges,

$C_0 \cong \mathbb{Z}^6$,

$C_1 \cong \mathbb{Z}^6$.

There are no nondegenerate 2-simplices, so

$C_2 = 0$.

Choose a cyclic orientation of the six edges. Let

$z = e_0 + e_1 + e_2 + e_3 + e_4 + e_5$.

The boundary terms cancel cyclically, giving

$\partial_1 z = 0$.

The graph is connected, so

$\operatorname{rank}(\operatorname{im} \partial_1) = 5$.

Therefore

$\operatorname{rank}(\ker \partial_1) = 6 - 5 = 1$.

Since $\operatorname{im} \partial_2 = 0$,

$H_1(N(\Omega\partial);\mathbb{Z}) \cong \mathbb{Z}$.

Connectedness gives

$H_0(N(\Omega\partial);\mathbb{Z}) \cong \mathbb{Z}$.

All higher homology groups vanish.

Thus

$H_n(N(\Omega\partial);\mathbb{Z})$
$\cong$
$\mathbb{Z}$, n=0,1,

and

$H_n(N(\Omega\partial);\mathbb{Z})=0$, $n\geq 2$.

The unique independent 1-cycle is represented by one traversal of the six-object boundary.

2.8 The Role of Selection

The deletion

$\Omega \mapsto \Omega\partial$

is not forced by the original seed.

The full nerve is contractible because the object A is a cone point. Removing A exposes the boundary carried by the remaining six objects.

This operation is therefore a declared topological policy:

retain the incidence world,

remove the universal lower object,

study the residual boundary.

The distinction is exact:

the seed derives the clone,

the bridge derives the meet,

the free policy constructs Ω,

the deletion policy selects Ω∂,

the nerve reveals S^1.

The circle is supported by the seed-generated architecture, but it is not an unavoidable invariant of the original operation alone.

2.9 Naturality Under Relabeling

Every permutation $\sigma \in S_3$ acts on subsets by

$S \mapsto \sigma(S)$.

This action preserves cardinality, nonemptiness, union, and reverse inclusion. Hence it defines an automorphism of Ω.

It fixes the full subset A and therefore restricts to an automorphism of Ω∂.

On the six-cycle, the action permutes the three singleton vertices and the three pair vertices while preserving incidence. The resulting symmetry group is the full dihedral symmetry group of the triangular boundary, represented through the natural action of S_3.

Thus the passage

seed presentation

→ free world

→ deleted category

→ six-cycle

commutes with relabeling.

The extracted circle is independent of the chosen names 0, 1, and 2.

2.10 First Topological Transition

The earlier development ended with a seven-object algebra reconstructed by three binary views.

The categorical extension gives

Ω

→ thin category Ω

→ nerve N(Ω)

→ contractible cone.

The selected deletion then gives

$\Omega\partial = \Omega - \{A\}$

$\rightarrow N(\Omega\partial) \cong C_6$

$\rightarrow |N(\Omega\partial)| \cong S^1$.

This is the first new transition of the present paper.

The self-indexing world is no longer only a finite algebraic object. Its proper-object incidence category carries the topology of the circle.

3. The Deleted Boundary and the Emergence of the Circle

3.1 The Proper-Object Subcategory

Let

$\Omega = P(A) - \{\emptyset\}$,

ordered by reverse inclusion, where

$A = \{0,1,2\}$.

The object A is initial in the associated thin category. Its presence makes the full nerve contractible.

Define the proper-object subcategory

$\partial\Omega := \Omega - \{A\}$.

Its objects are precisely the nonempty proper subsets of A:

{0}, {1}, {2},

{0,1}, {0,2}, {1,2}.

A morphism exists from S to T exactly when

$S \supseteq T$.

Since every object has cardinality one or two, every nonidentity morphism runs from a two-element subset to one of its singleton members.

Thus ∂Ω has six nonidentity arrows:

{0,1} → {0},

{0,1} → {1},

{0,2} → {0},

{0,2} → {2},

{1,2} → {1},

{1,2} → {2}.

There are no composable pairs of nonidentity arrows.

3.2 The Incidence Graph

The nerve N(∂Ω) is therefore one-dimensional.

Its vertices are the six objects of ∂Ω. Its edges are the six incidence morphisms.

Choose the cyclic ordering

{0}
—
{0,1}
—
{1}
—
{1,2}
—
{2}
—
{0,2}
—
{0}.

Every vertex has degree two. The graph is connected and has six vertices and six edges. Hence it is the cycle graph

C_6.

Therefore,

$N(\partial\Omega) \cong C_6$.

The graph is bipartite. One part consists of singleton subsets, and the other consists of two-element subsets. The bipartition records the distinction between vertices and edges of an underlying triangle.

3.3 Barycentric Subdivision

Let Δ^2 be the abstract 2-simplex on vertex set A.

Its boundary $\partial\Delta^2$ contains:

three vertices,

three edges,

and no 2-simplex.

The nonempty faces of $\partial\Delta^2$ are exactly the six objects of $\partial\Omega$.

The face poset of $\partial\Delta^2$ is ordered by inclusion. The category $\partial\Omega$ uses reverse inclusion, but reversing all arrows does not change the geometric realization of the nerve.

The nerve of the face poset is the barycentric subdivision. Hence

$N(\partial\Omega) \cong Sd(\partial\Delta^2)$

up to categorical opposition.

Geometric realization commutes with barycentric subdivision up to canonical homeomorphism. Therefore,

$|N(\partial\Omega)| \cong |\partial\Delta^2|$.

Since

$|\partial\Delta^2| \cong S^1$,

it follows that

$|N(\partial\Omega)| \cong S^1$.

The circle arises as the order complex of the nonempty proper subsets of a three-element set.

3.4 A General Boundary Formula

For a finite set A with $|A| = n+1$, the poset of nonempty proper subsets of A is the proper part of the Boolean lattice.

Its order complex is the barycentric subdivision of the boundary of the n-simplex:

$N(P(A) - \{\emptyset, A\}) \cong Sd(\partial\Delta^n)$.

Therefore,

$|N(P(A) - \{\emptyset, A\})| \cong S^{n-1}$.

In the present case,

$|A| = 3$,

so

$n = 2$

and

$S^{n-1} = S^1$.

Thus the emergence of the circle is the first nontrivial case of the general Boolean-boundary identity

proper Boolean lattice on n+1 generators

→ sphere S^{n-1}.

The three-state seed supplies the index set required for the one-dimensional case.

3.5 Simplicial Chain Complex

Choose vertices

$v_0 = \{0\}$,

$v_1 = \{0,1\}$,

$v_2 = \{1\}$,

$v_3 = \{1,2\}$,

$v_4 = \{2\}$,

$v_5 = \{0,2\}$.

Orient the edges cyclically:

$e_i = [v_i, v_{i+1}]$,

with indices modulo 6.

The simplicial chain groups are

$C_1 \cong \mathbb{Z}^6$,

$C_0 \cong \mathbb{Z}^6$,

$C_n = 0$ for $n \geq 2$.

The boundary map is

$\partial_1 e_i = v_{i+1} - v_i$.

For a general 1-chain

$c = \Sigma_i \, a_i e_i$,

the coefficient of v_i in $\partial_1 c$ is

$a_{i-1} - a_i$.

Thus

$\partial_1 c = 0$

if and only if

$a_0 = a_1 = \cdots = a_5$.

Consequently,

$\ker \partial_1 = \mathbb{Z}z$,

where

$z = e_0 + e_1 + e_2 + e_3 + e_4 + e_5$.

Because $C_2 = 0$,

$\operatorname{im} \partial_2 = 0$.

Hence

$H_1(N(\partial\Omega);\mathbb{Z}) = \ker \partial_1 \cong \mathbb{Z}$.

Connectedness gives

$H_0(N(\partial\Omega);\mathbb{Z}) \cong \mathbb{Z}$.

Therefore,

$H_k(N(\partial\Omega);\mathbb{Z})$
$\cong$
$\mathbb{Z}$, k=0,1,

and

$H_k(N(\partial\Omega);\mathbb{Z})=0$, $k\geq 2$.

3.6 Euler Characteristic

The Euler characteristic is

$\chi = f_o - f_1$

with

$f_o = 6$,

$f_1 = 6$.

Therefore,

$\chi(N(\partial\Omega)) = 6 - 6 = 0$.

For a connected finite graph,

rank $H_1 = 1 - \chi$.

Thus

rank $H_1 = 1$.

The Euler characteristic, chain complex, and geometric realization independently identify the same topology.

3.7 Orientation and Fundamental Cycle

The simplicial complex does not contain a preferred orientation. An orientation must be selected.

Once a cyclic ordering is chosen, the fundamental 1-cycle is

$z = \Sigma_i e_i$.

Reversing the orientation replaces z by −z.

Thus the first homology group is canonical, while a particular generator is determined only up to sign:

$H_1(S^1;\mathbb{Z}) \cong \mathbb{Z}$,

$[z] \mapsto \pm 1$.

Every permutation of A induces an automorphism of the six-cycle. Even permutations preserve one cyclic orientation, while odd permutations reverse it.

The relabeling action on first homology is therefore the sign representation:

$S_3 \to \mathrm{Aut}(\mathbb{Z})$,

$\sigma \mapsto$ multiplication by $\mathrm{sgn}(\sigma)$.

This is the first point at which relabeling symmetry acts nontrivially on a derived topological invariant.

3.8 From Meet Structure to Incidence Topology

The path to the circle can be compressed as

(A,w)

→ derived meet algebra B

→ free nonempty meet-semilattice Ω

→ proper part ∂Ω

→ order complex N(∂Ω)

→ $\mathrm{Sd}(\partial\Delta^2)$

→ S^1.

The first three transitions are algebraic. The fourth is a selected deletion. The fifth converts order into simplicial incidence. The sixth identifies a standard subdivision. The seventh recognizes the resulting topological type.

No metric, coordinate embedding, or continuous equation for the circle is supplied. The topology arises entirely from finite comparability relations.

3.9 What the Circle Does and Does Not Mean

The circle is not the homotopy type of the original seed algebra.

The full seven-object nerve is contractible:

$|N(\Omega)| \simeq *$.

The circle is the homotopy type of the proper-object boundary:

$|N(\partial\Omega)| \simeq S^1$.

Thus the valid statement is

the seed-supported free world contains a canonically recognizable Boolean boundary whose realization is a circle.

It would be too strong to state

the seed itself is a circle

or

functional completeness forces S^1.

The circle depends on the selected free-world construction and the removal of its cone point.

3.10 Boundary Theorem

The result may be stated tersely.

Theorem. Let A be the three-element carrier of the seed and let

$\Omega := F^+\Lambda(A) \cong P(A) - \{\emptyset\}$.

Regard Ω as a thin category under reverse inclusion, and remove its least object A. Then

$N(\Omega - \{A\}) \cong Sd(\partial\Delta^2)$

and therefore

$|N(\Omega - \{A\})| \cong S^1$.

Moreover,

$H_0 \cong \mathbb{Z}$,

$H_1 \cong \mathbb{Z}$,

$H_n = 0$ for $n \geq 2$,

and the S_3 relabeling action on H_1 is multiplication by the permutation sign.

The finite self-indexing world therefore contains a certified one-dimensional homotopy boundary. Its next structure is determined by the universal covering of that circle.

4. Universal Covers, Winding, and the Fundamental Groupoid

4.1 The Six-Cycle as a Combinatorial Circle

Let

$X := N(\partial\Omega) \cong C_6$.

Choose a cyclic ordering of the vertices

$v_0, v_1, v_2, v_3, v_4, v_5$,

with indices taken modulo 6.

Let e_i denote the oriented edge from v_i to v_{i+1}. The reverse orientation is denoted e_i^{-1}.

A combinatorial path in X is a finite sequence of oriented edges whose endpoints match. Path composition is concatenation. Reversal changes the order of the edges and reverses each orientation.

The topology of X is completely encoded by this finite path calculus.

4.2 The Integer-Line Cover

Define an infinite graph $\tilde{X}$ with vertices indexed by the integers:

$\mathrm{Vert}(\tilde{X}) = \mathbb{Z}$.

Place one edge between each pair of consecutive integers:

$n \to n+1$.

Define the projection

$p: \tilde{X} \to X$

on vertices by

$p(n) = v_n \bmod 6$.

Each edge

$n \to n+1$

maps to

$e_n \bmod 6$.

Thus $\tilde{X}$ is the integer line, periodically projected onto the six-cycle.

Every vertex n has exactly two incident edges, and p maps its local star bijectively onto the local star of v_n mod 6. Therefore p is a graph covering.

The covering is connected and $\tilde{X}$ contains no cycles. Hence $\tilde{X}$ is a tree and therefore simply connected.

Thus

$p: \tilde{X} \to X$

is the universal cover.

Geometrically,

$|\tilde{X}| \cong \mathbb{R}$

and

$|X| \cong S^1$.

4.3 Unique Path Lifting

Let

$\gamma = a_1 a_2 \cdots a_k$

be an edge path in X, where each a_i is an oriented edge.

Choose a lift $n_o \in \mathbb{Z}$ of the initial vertex of γ.

At each step there is exactly one edge leaving the current lifted vertex that projects to the required oriented edge in X. Therefore γ has a unique lift

$\tilde{\gamma}$

starting at n_o.

If γ is closed, its lift need not be closed. Its endpoint differs from its initial point by an integer multiple of 6:

$n_k - n_o = 6q$

for a unique $q \in \mathbb{Z}$.

This integer measures how many times γ winds around the cycle.

4.4 Winding Number

For an oriented edge a in X, define

$\varepsilon(a)=1$

when a follows the chosen cyclic orientation, and

$\varepsilon(a)=-1$

when it opposes that orientation.

For a path

$\gamma = a_1 a_2 \cdots a_k$,

define its signed length

$L(\gamma)=\Sigma_i\, \varepsilon(a_i)$.

If γ is closed, then $L(\gamma)$ is divisible by 6. Define the winding number

$\mathrm{wind}(\gamma)=L(\gamma)/6$.

Equivalently, if $\tilde{\gamma}$ begins at n_o and ends at n_k, then

$\mathrm{wind}(\gamma)=(n_k-n_o)/6$.

The winding number is additive under concatenation:

$\mathrm{wind}(\gamma\delta)=\mathrm{wind}(\gamma)+\mathrm{wind}(\delta)$.

It changes sign under reversal:

$\mathrm{wind}(\gamma^{-1})=-\mathrm{wind}(\gamma)$.

The positively oriented fundamental cycle has winding number 1.

4.5 Backtrack Reduction

An immediate reversal

aa^{-1}

contributes zero signed length and can be deleted without changing endpoints or winding number.

Repeated deletion of adjacent inverse pairs reduces every path to a backtrack-free path.

In the integer-line cover, a reduced path is monotone. Therefore a reduced closed path in X is determined by how many complete circuits it makes and in which direction.

Two closed paths based at the same vertex are homotopic exactly when they have the same winding number.

Hence winding gives a complete invariant of based loop homotopy.

4.6 The Fundamental Group

Fix a base vertex v_o.

Let [γ] denote the homotopy class of a loop at v_o. Define

$\Phi: \pi_1(X,v_o) \to \mathbb{Z}$

by

$\Phi([\gamma])=\text{wind}(\gamma)$.

This map is a group homomorphism because winding is additive under concatenation.

It is surjective because the fundamental six-edge cycle has winding number 1.

It is injective because a loop of winding number zero lifts to a closed path in the tree $\tilde{X}$ and is therefore null-homotopic.

Thus

$\pi_1(X,v_o) \cong \mathbb{Z}$.

The homotopy class of one positively oriented circuit is a generator.

The inverse generator is represented by the same cycle traversed in the opposite direction.

4.7 Deck Transformations

For each $k \in \mathbb{Z}$, define

$T_k: \tilde{X} \to \tilde{X}$

by

$T_k(n)=n+6k$.

Then

$p(T_k(n))=p(n)$,

so T_k is a deck transformation.

Every deck transformation is determined by the image of one vertex. Since it must preserve fibers and adjacency, it has the form T_k for a unique k.

Therefore,

$\text{Deck}(p) \cong \mathbb{Z}$.

Composition satisfies

$T_k T_l = T_{k+l}$.

The standard identification gives

$\pi_1(X, v_0) \cong \mathrm{Deck}(p) \cong \mathbb{Z}$.

A loop of winding number k lifts from a chosen point n to a path ending at $T_k(n)$.

4.8 Fibers as Torsors

For each vertex v_i of X, the fiber is

$p^{-1}(v_i) = \{i+6k : k \in \mathbb{Z}\}$.

The deck group acts freely and transitively on this fiber.

Thus every fiber is a $\mathbb{Z}$-torsor. It has no preferred origin until one lift is selected.

This reflects a general distinction:

the group $\mathbb{Z}$ describes relative displacement,

while a fiber records possible lifted positions without a canonical zero.

The universal cover therefore carries both group structure in its symmetries and torsor structure in its fibers.

4.9 The Fundamental Groupoid

The based fundamental group records loops at one selected vertex. The fundamental groupoid retains all six vertices.

Define $\Pi_1(X)$ with:

objects: $v_0, \ldots, v_5$,

morphisms: homotopy classes of paths between vertices.

Choose integer representatives $i, j \in \{0, \ldots, 5\}$. A lifted path from v_i to v_j beginning at i ends at

$j+6k$

for a unique $k \in \mathbb{Z}$.

Therefore a morphism from v_i to v_j can be encoded by an integer k:

$\mathrm{Hom}_{\Pi_1(X)}(v_i, v_j) \cong \mathbb{Z}$.

Write such a morphism as

(i,j;k).

Its lifted endpoint displacement is

(j−i)+6k.

Composition is

(j,ℓ;m) ∘ (i,j;k)

(i,ℓ;k+m).

The identity is

(i,i;0).

The inverse is

$(i,j;k)^{-1}=(j,i;-k)$.

Thus $\Pi_1(X)$ is an explicit arithmetic groupoid.

Every automorphism group is

$\mathrm{Aut}\Pi_1(X)(v_i) \cong \mathbb{Z}$.

All six objects are isomorphic, and the groupoid is connected.

4.10 Reduction to One Object

Choose the base object v_o. Since $\Pi_1(X)$ is connected, it is equivalent to the one-object groupoid whose automorphism group is $\mathbb{Z}$:

$\Pi_1(X) \simeq B\mathbb{Z}$.

Here $B\mathbb{Z}$ denotes the category with one object and endomorphism group $\mathbb{Z}$.

Passing from $\Pi_1(X)$ to $B\mathbb{Z}$ forgets the six chosen vertices but preserves the complete homotopy 1-type of the circle.

Thus

X

$\to \Pi_1(X)$

$\to B\mathbb{Z}$

is a categorical compression of the circle.

The first object retains the finite simplicial presentation. The second retains all points and path classes. The third retains only the loop group up to equivalence.

4.11 Relabeling and Orientation

A permutation $\sigma \in S_3$ acts on X by a symmetry of the six-cycle.

Even permutations preserve the selected orientation. Odd permutations reverse it.

The induced action on the universal cover may therefore have the form

$n \mapsto n+c$

for orientation-preserving symmetries, or

$n \mapsto -n+c$

for orientation-reversing symmetries.

On the deck group and fundamental group, the induced automorphism is

$k \mapsto \mathrm{sgn}(\sigma)k$.

Hence the relabeling action on

$\pi_1(X) \cong \mathbb{Z}$

is again the sign representation.

The fundamental group is independent of the labels, but the choice of positive generator is not.

4.12 Covering-Theoretic Transition

The deleted boundary supplied the circle:

$\partial\Omega$

$\rightarrow X \cong C_6$

$\rightarrow |X| \cong S^1$.

Covering theory now supplies

$\tilde{X} \cong \mathbb{Z}$-line,

$\pi_1(X) \cong \mathbb{Z}$,

$\mathrm{Deck}(\tilde{X}/X) \cong \mathbb{Z}$,

$\Pi_1(X) \simeq B\mathbb{Z}$.

The finite six-object boundary has therefore generated an infinite arithmetic symmetry.

A single circuit in the finite graph becomes translation by six units in the universal cover. Iterated circuits become the additive group of integers.

This is the second major topological transition:

finite incidence

→ cyclic path structure

→ universal cover

→ integer-valued winding

→ fundamental groupoid.

The next stage uses finite-index subgroups of $\mathbb{Z}$ to construct cyclic covers and uses representations of $\mathbb{Z}$ to classify monodromy and local coefficient systems.

5. Cyclic Covers, Monodromy, and Integral Local Systems

5.1 Finite Cyclic Covers

Let

$X \cong C_6$

be the six-cycle obtained from the proper-object boundary.

Its universal cover has vertex set $\mathbb{Z}$ and covering projection

$p(n)=v_n \bmod 6$.

For each integer $q \geq 1$, quotient the universal cover by the subgroup

$q\mathbb{Z} \leq \mathrm{Deck}(p) \cong \mathbb{Z}$.

Equivalently, identify

$n \sim n+6q$.

The resulting finite graph Xq has

6q

vertices arranged in a cycle.

The covering map

$p_q: X_q \to X$

is defined by reduction modulo 6.

Each vertex of X has exactly q preimages. Therefore p_q is a q-sheeted connected covering.

The cases begin with

$q=1$: $X_1 \cong C_6$,

$q=2$: $X_2 \cong C_{12}$,

$q=3$: $X_3 \cong C_{18}$.

In general,

$X_q \cong C_{6q}$.

Thus the finite connected covers of the seed-generated circle are indexed by positive integers.

5.2 Subgroups of the Fundamental Group

Connected coverings of X correspond to subgroups of

$\pi_1(X) \cong \mathbb{Z}$.

Every subgroup of $\mathbb{Z}$ has the form

$q\mathbb{Z}$

for a unique $q \geq 0$.

The subgroup $q\mathbb{Z}$ corresponds to the q-sheeted cyclic cover when $q \geq 1$.

The trivial subgroup

$\{0\}$

corresponds to the universal cover.

The full subgroup

$\mathbb{Z}$

corresponds to the identity covering.

Hence the covering theory is completely classified by divisibility:

$q\mathbb{Z} \leq r\mathbb{Z}$

exactly when

r divides q.

A covering X_q factors through X_r precisely when r divides q.

The finite covering lattice is therefore the divisibility lattice of positive integers.

5.3 Monodromy of a Cover

Choose a base vertex v_o in X and label the q points in its fiber by

0,1,…,q−1.

Let γ be the positively oriented generator of $\pi_1(X,v_o)$.

Lifting γ from fiber point j ends at

j+1 mod q.

Thus the monodromy permutation is the q-cycle

$\tau_q=(0\ 1\ 2\ \cdots\ q-1)$.

The monodromy representation is

$\rho_q: \mathbb{Z} \to S_q$

with

$\rho_q(k)=\tau_q^k$.

The image acts transitively on the fiber, reflecting connectedness of the cover.

Every connected q-sheeted cover of the circle has monodromy generated by a transitive q-cycle. Up to relabeling of the fiber, this is unique.

The complete finite covering data are therefore compressed into one permutation.

5.4 Local Systems

A local system of abelian groups on X is a functor

$L: \Pi_1(X) \to Ab$.

Because

$\Pi_1(X) \simeq B\mathbb{Z}$,

a local system is equivalent to an abelian group M equipped with an automorphism

$T: M \to M$.

The group M is the fiber over a selected base point. The automorphism T is the monodromy around one positive circuit.

Thus

$\mathrm{LocAb}(X) \simeq \mathrm{Rep}(\mathbb{Z},\mathrm{Ab})$.

For integral free modules of finite rank, an object is a pair

(M,T)

with

$M \cong \mathbb{Z}^r$

and

$T \in \mathrm{GLr}(\mathbb{Z})$.

The infinite path structure of the circle is encoded by powers of one invertible matrix:

$k \mapsto T^k$.

5.5 Rank-One Integral Local Systems

For

$M=\mathbb{Z}$,

the only automorphisms are multiplication by

$+1$

or

-1.

Hence there are two rank-one integral local systems up to isomorphism.

The trivial system has

$T=1$.

The sign system has

$T=-1$.

The trivial system returns every element unchanged after one circuit.

The sign system reverses every element after one circuit.

These are the simplest possible examples of global twisting over the seed-generated circle.

5.6 Cellular Cohomology with Local Coefficients

Give X one 0-cell and one 1-cell.

For a local system determined by

$T \in \mathrm{Aut}(M)$,

the cellular cochain complex is

$0 \to M \to M \to 0$,

with coboundary

$\delta = T - I$.

Therefore,

$H^0(X;M_T)=\ker(T-I)$,

and

$H^1(X;M_T)=\mathrm{coker}(T-I)$.

All higher cohomology groups vanish.

Thus

H^0

measures monodromy-invariant elements, while

H^1

measures the failure of $T-I$ to be surjective.

For the trivial rank-one system,

$T=1$,

so

$T-I=0$.

Hence

$H^0(X;\mathbb{Z}) \cong \mathbb{Z}$,

$H^1(X;\mathbb{Z}) \cong \mathbb{Z}$.

For the sign system,

$T=-1$,

so

$T-I=-2$.

Hence

$H^0(X;\mathbb{Z}_-)=0$,

$H^1(X;\mathbb{Z}_-) \cong \mathbb{Z}/2\mathbb{Z}$.

The same underlying circle therefore supports different cohomology according to the chosen transport law.

5.7 Matrix Monodromy

Let

$M=\mathbb{Z}^r$

and let

$T \in GLr(\mathbb{Z})$.

Then

$H^0(X;M_T)=\ker(T-I)$,

and

$H^1(X;M_T)=\mathbb{Z}^r/(T-I)\mathbb{Z}^r$.

The structure of H^1 is obtained from the Smith normal form of $T-I$.

Suppose

$U(T-I)V = \mathrm{diag}(d_1,\ldots,d_s,0,\ldots,0)$,

where each nonzero diagonal entry satisfies

d_i divides d_{i+1}.

Then

$H^1(X;M_T)$
$\cong$
$\mathbb{Z}^{r-s}$

$\oplus$
$\mathbb{Z}/d_1\mathbb{Z}$
$\oplus \cdots \oplus$
$\mathbb{Z}/d_s\mathbb{Z}$.

The free rank records invariant directions not reached by T−I. The torsion factors record finite obstructions to solving

(T−I)x=y.

Thus a finite integer matrix converts the circle into explicit arithmetic invariants.

5.8 Examples

Let

T=
[1 1
0 1].

Then

T−I=
[0 1
0 0].

Its kernel is generated by

(1,0),

so

$H^0 \cong \mathbb{Z}$.

Its cokernel is also free of rank one, so

$H^1 \cong \mathbb{Z}$.

Now let

T=
[0 −1
1 0].

This matrix has order four.

Then

$T - I =$

$\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$.

Its determinant is

2.

The kernel is zero, and the cokernel has order two. Therefore,

$H^0 = 0$,

$H^1 \cong \mathbb{Z}/2\mathbb{Z}$.

These examples show that identical base topology can support distinct fixed submodules and torsion quotients.

5.9 The Representation Category

Define

$\mathrm{Rep}_{\mathbb{Z}}(\mathbb{Z})$

to be the category whose objects are pairs

(M,T),

where M is a finitely generated free abelian group and T is an automorphism of M.

A morphism

$f: (M,T) \to (N,S)$

is a homomorphism satisfying

$fT = Sf$.

Thus morphisms are intertwiners.

Composition is ordinary composition of homomorphisms. The intertwining condition is preserved:

$gS = T'g$

and

$fT = Sf$

imply

$gfT=T'gf$.

The identity map is an intertwiner.

Hence $\mathrm{Rep}_{\mathbb{Z}}(\mathbb{Z})$ is a well-defined category.

Through the fundamental groupoid equivalence,

$\mathrm{Loc}_{\mathbb{Z}}(X) \simeq \mathrm{Rep}_{\mathbb{Z}}(\mathbb{Z})$.

The finite six-cycle therefore supports an entire category of arithmetic transport systems.

5.10 Direct Sums and Tensor Products

Given two representations

(M,T)

and

(N,S),

their direct sum is

$(M \oplus N, T \oplus S)$.

Their tensor product is

$(M \otimes N, T \otimes S)$.

The unit object for tensor product is

$(\mathbb{Z},1)$.

Dualization sends

(M,T)

to

$(M^*,(T^{-1})^*)$.

Thus the representation category inherits standard additive and monoidal structure.

The rank-one sign system is tensor-invertible because

$$(\mathbb{Z},-1) \otimes (\mathbb{Z},-1) \cong (\mathbb{Z},1).$$

This gives a nontrivial element of order two among rank-one invertible local systems.

5.11 Relabeling and Monodromy Inversion

An odd permutation of A reverses the orientation of the six-cycle.

Orientation reversal sends the generator

$1 \in \pi_1(X)$

to

-1.

Therefore a monodromy representation determined by T is transported to one determined by

T^{-1}.

The relabeling action on local systems is

$(M,T) \mapsto (M,T^{-1})$

for orientation-reversing permutations.

For rank-one systems,

$1^{-1}=1$

and

$(-1)^{-1}=-1$,

so both rank-one integral systems are fixed.

For higher-rank systems, T and T^{-1} need not be conjugate. Relabeling can therefore exchange distinct local systems.

The seed symmetry acts not only on the base circle but also on its transport categories.

5.12 From Covering Theory to Representation Theory

The progression is now

$\pi_1(X) \cong \mathbb{Z}$

→ subgroups $q\mathbb{Z}$

→ cyclic covers

→ permutation monodromy

→ module monodromy

→ integral local systems

→ Rep$\mathbb{Z}(\mathbb{Z})$.

Finite covers use actions of $\mathbb{Z}$ on finite sets.

Integral local systems use actions of $\mathbb{Z}$ on free abelian groups.

Both are manifestations of the same principle:

a structure over the circle is determined by the effect of one fundamental circuit.

The finite seed-generated boundary therefore supports a passage from topology to arithmetic representation theory.

5.13 Scope of the Extension

The original seed does not uniquely select a coefficient module M or a monodromy automorphism T.

These are additional construction choices.

What is derived is the base circle and its fundamental group

$\pi_1(X) \cong \mathbb{Z}$.

Once a target category is selected, representations of $\mathbb{Z}$ classify transport around that circle.

The logical structure is therefore

seed-supported boundary

$\vdash \pi_1(X) \cong \mathbb{Z}$,

choice of M and T

defines a local system,

T−I

determines its cohomology.

The topology supplies the universal transport law. The coefficient system supplies the arithmetic content.

The next branch returns to the original seed and promotes its six relabelings from consistency checks to the objects and arrows of a moduli groupoid.

6. Relabeling Symmetry and the Seed Action Groupoid

6.1 Relabeled Presentations of the Seed

Let

$G = S_3$

act on the carrier

$A = \{0,1,2\}$

by permutations.

For $\sigma \in S_3$, define the relabeled operation

w^σ(x,y)

$\sigma(w(\sigma^{-1}x,\sigma^{-1}y))$.

The operation w^σ is the same abstract algebraic structure written in the labels transported by σ.

There are six permutations of A. Applying all six to w produces the orbit

Orb(w)

$\{w^\sigma : \sigma \in S_3\}$.

The orbit contains six distinct binary tables.

Thus

$|Orb(w)| = 6$.

No two different permutations produce the same labeled operation.

6.2 The Stabilizer

The stabilizer of w is

Stab(w)

$\{\sigma \in S_3 : w^\sigma = w\}$.

Equivalently, σ lies in Stab(w) exactly when

σ(w(x,y))

w(σx,σy)

for all $x,y \in A$.

This is the automorphism group of the algebra (A,w):

$\mathrm{Aut}(A,w)=\mathrm{Stab}(w)$.

Exhaustive evaluation over all six permutations gives

$\mathrm{Aut}(A,w)=1$.

Only the identity permutation preserves the operation table.

The seed is therefore rigid.

Rigidity does not mean that the carrier has no permutations. The bare set A has symmetry group S_3. Rigidity means that none of the five nonidentity permutations preserves w.

Thus

$\mathrm{Aut}(A)=S_3$,

while

$\mathrm{Aut}(A,w)=1$.

The distinction between carrier symmetry and structure-preserving symmetry is central to the moduli analysis.

6.3 Orbit Size

The orbit–stabilizer formula gives

$|\mathrm{Orb}(w)|$

$|S_3| \,/\, |\mathrm{Aut}(A,w)|$.

Since

$|S_3|=6$

and

$|\mathrm{Aut}(A,w)|=1$,

it follows that

$|\mathrm{Orb}(w)|=6$.

The six labeled tables are not six different abstract algebras. They form one orbit under relabeling.

Each table is a presentation of the same isomorphism class.

The orbit records coordinate variation. The stabilizer records internal symmetry.

For this seed, coordinate variation is maximal and internal symmetry is trivial.

6.4 The Action Groupoid

The relabeling action determines an action groupoid

$\mathcal{G} := S_3 \ltimes \mathrm{Orb}(w)$.

Its objects are the six labeled seed presentations.

A morphism

$u \to v$

is a permutation $\sigma \in S_3$ satisfying

$\sigma \cdot u = v$.

Composition is multiplication in S_3.

If

$u \to v$

is represented by σ and

$v \to z$

is represented by τ, then the composite

$u \to z$

is represented by

$\tau\sigma$.

The identity morphism at u is represented by the identity permutation.

Every morphism is invertible, with inverse represented by σ^{-1}.

Hence $\mathcal{G}$ is a groupoid.

6.5 Exact Arrow Counts

Because the action is free and transitive, for every ordered pair of objects u,v there is exactly one permutation σ with

$\sigma \cdot u = v$.

Therefore,

$|\mathrm{Hom}_{\mathcal{G}}(u,v)|=1$

for all $u,v \in \mathrm{Orb}(w)$.

Since there are six objects, the total number of arrows is

$6 \times 6 = 36$.

The groupoid is connected.

Each automorphism group is trivial:

$\mathrm{Aut}_{\mathcal{G}}(u)=1$.

Thus $\mathcal{G}$ is a connected thin groupoid.

As an ordinary category, it has one unique isomorphism between every ordered pair of objects.

6.6 Relabeling as Morphism Rather Than Test

In the earlier construction, relabeling was used to check that the developmental route commuted with permutations of A.

The action groupoid strengthens that observation.

A relabeling is no longer only an external comparison between two executions. It becomes an internal morphism between two presentations.

The six tables are objects.

The six carrier permutations generate arrows.

Composition of relabelings becomes categorical composition.

Inverse relabeling becomes categorical inversion.

The statement

labels do not change the mathematics

is therefore represented by an explicit connected groupoid.

6.7 Transport of Derived Terms

Each relabeling transports every term-defined operation.

If t is a term in the language of w, then its interpretation under the relabeled seed satisfies

$t^{w^\sigma}(x_1,\dots,x_n)$

σ
$t^w(\sigma^{-1}x_1,\dots,\sigma^{-1}x_n)$.

In particular, the retraction transports as

r^σ

$\sigma\, r\, \sigma^{-1}$.

Its image is

B^σ

$\sigma(B)$.

The derived meet transports as

$m^\sigma(x,y)$

σ
$m(\sigma^{-1}x,\sigma^{-1}y)$.

Thus every object of the action groupoid carries its own relabeled copy of the two-element meet algebra.

The bridge is natural across all arrows of $\mathcal{G}$.

6.8 Transport of the Free World

The free world associated with a presentation u is

$\Omega_u := F^+\Lambda(A_u)$.

A permutation σ induces

$\Omega_u \to \Omega_{\sigma\cdot u}$

by

$S \mapsto \sigma(S)$.

This map preserves nonemptiness, union, and reverse inclusion.

Hence it is an isomorphism of meet-semilattices and thin categories.

The coordinate views also transport:

$v_{\sigma(a)}(\sigma(S)) = v_a(S)$

after identifying the relabeled two-element algebra with the original one.

Thus the entire chain

seed

→ retraction

→ meet

→ free world

→ coordinate reconstruction

extends functorially across the action groupoid.

6.9 Transport of the Boundary Circle

The full subset A is fixed setwise by every permutation, so relabeling restricts to the proper-object boundary.

Each $\sigma \in S_3$ induces an automorphism

$\partial\Omega \to \partial\Omega$.

On the six-cycle, this action permutes singleton objects and pair objects while preserving incidence.

Even permutations preserve a selected orientation of the cycle.

Odd permutations reverse it.

Therefore the induced action on

$H_1(S^1;\mathbb{Z}) \cong \mathbb{Z}$

is

$n \mapsto \mathrm{sgn}(\sigma)n$.

The same action sends a monodromy matrix T to T or T^{-1} according to the parity of σ.

Thus the relabeling groupoid acts coherently on the algebraic, categorical, topological, and representation-theoretic layers.

6.10 The Moduli Interpretation

The action groupoid $\mathcal{G}$ is a finite moduli object for labeled realizations of the seed.

Its objects answer:

Which labeled tables present the seed?

Its arrows answer:

Which relabelings identify one presentation with another?

Its automorphism groups answer:

Which relabelings preserve a presentation?

For the present seed:

there are six presentations,

all six are mutually isomorphic,

each has trivial automorphism group.

Thus the moduli problem has one isomorphism class and no nontrivial isotropy.

This distinction cannot be represented adequately by the orbit set alone.

The orbit set records one class.

The groupoid records both the class and all explicit coordinate identifications among its presentations.

6.11 Comparison with the Bare Carrier

The action of S_3 on the bare carrier has one object up to isomorphism but a nontrivial automorphism group.

Its one-object groupoid is

BS_3.

The seed algebra instead has automorphism group

Aut(A,w)=1.

Its one-object moduli skeleton is therefore

B1.

These two classifying groupoids are not equivalent:

$BS_3 \not\simeq B1$.

The operation destroys the internal symmetry of the carrier.

Thus the seed does more than generate operations. It selects a rigid structure from a maximally symmetric three-element set.

The contrast is

bare carrier: six internal symmetries,

structured seed: one internal symmetry.

Here the unique symmetry is the identity.

6.12 Rigidity as Information

A rigid algebra has no nontrivial self-identifications.

Every labeled presentation is connected to every other by a unique relabeling, but no presentation has a nonidentity loop.

The six-object groupoid therefore has many arrows between distinct objects but no nontrivial automorphisms.

This pattern is characteristic of a free transitive group action.

The apparent multiplicity of six tables is entirely presentational.

After quotienting by isomorphism, one object remains.

After retaining automorphisms, no additional loop structure remains.

The moduli groupoid is therefore contractible.

6.13 Action-Groupoid Theorem

The result may be stated compactly.

Theorem. Let S_3 act by conjugate relabeling on the orbit of the seed w. Then:

1. Orb(w) has six elements.
2. Aut(A,w)=1.
3. The action is free and transitive.
4. The action groupoid

$S_3 \ltimes \mathrm{Orb}(w)$

has six objects and 36 arrows.

5. There is exactly one arrow between every ordered pair of objects.

6. Every isotropy group is trivial.
7. The groupoid is connected and equivalent to the terminal groupoid.

Thus the six relabeled seed tables form one rigid moduli point.

6.14 The New Categorical Branch

The algebraic-topological branch began with the free world and extracted a circle.

The moduli branch begins directly from the seed:

(A,w)

→ S_3-action on labeled presentations

→ six-object action groupoid

→ trivial stabilizer

→ one rigid isomorphism class.

The next section constructs the orbit–stabilizer equivalence explicitly and identifies the nerve of the action groupoid as a homotopy quotient and classifying space.

7. Orbit–Stabilizer Equivalence and Homotopy Quotients

7.1 The General Action Groupoid

Let a finite group G act on a set X.

The action groupoid

$G \ltimes X$

has:

objects $x \in X$,

morphisms $x \to y$ given by elements $g \in G$ satisfying

$g \cdot x = y$.

Composition is inherited from multiplication in G.

For $x \in X$, the automorphism group of x is its stabilizer:

$Aut(x)=G_x$

with

$G_x=\{g\in G : g\cdot x=x\}$.

The connected components of $G \ltimes X$ are the G-orbits in X.

Thus an action groupoid retains both:

the quotient information carried by X/G,

and the isotropy information carried by the stabilizers.

A quotient set discards the stabilizers. The action groupoid preserves them.

7.2 The Transitive Case

Suppose the action of G on X is transitive.

Choose a base object $x_o \in X$ and let

$H=G_{x_o}$

be its stabilizer.

Every object $x \in X$ can be written as

$x=g\cdot x_o$

for some $g \in G$.

The orbit is therefore identified with the coset space

G/H.

The action groupoid

$G \ltimes X$

is connected, and every automorphism group is conjugate to H.

Its entire equivalence type is determined by the stabilizer of one object.

This gives the groupoid form of orbit–stabilizer:

$G \ltimes X \simeq BH$,

where BH is the one-object groupoid with automorphism group H.

7.3 Explicit Choice of Representatives

For each $x \in X$, choose an element

$s_x \in G$

such that

$s_x \cdot x_o = x$.

Require

$s_{xo} = e$.

These representatives define arrows

$s_x : x_o \to x$

in the action groupoid.

Given a morphism

$g : x \to y$,

define

$F(g) = s_y^{-1} g s_x$.

Since

$s_x \cdot x_o = x$

and

$g \cdot x = y$,

we obtain

$s_y^{-1} g s_x \cdot x_o = x_o$.

Therefore,

$F(g) \in H$.

Define a functor

$F : G \ltimes X \to BH$

by sending every object to the unique object of BH and every arrow $g \to y$ to

$s_y^{-1} g s_x$.

7.4 Functoriality

For the identity arrow at x,

$F(id_x)$

$s_x^{-1} e s_x$

e.

Thus identities are preserved.

Let

g: x → y

and

k: y → z.

Then

F(kg)

$s_z^{-1} k g s_x$.

Insert

$s_y s_y^{-1}$

between k and g:

F(kg)

$(s_z^{-1} k s_y)(s_y^{-1} g s_x)$.

Hence

F(kg)=F(k)F(g).

Therefore F is a functor.

7.5 Inclusion of the Stabilizer

Define a functor

I: BH → G ⋉ X

by sending the unique object of BH to x_o and sending h ∈ H to the corresponding automorphism

h: x_o → x_o.

Then

F∘I = id_BH.

The composite

$I \circ F$

sends every object x to x_o.

It is not equal to the identity functor on $G \ltimes X$, but it is naturally isomorphic to it.

For each x, use the chosen arrow

$s_x: x_o \to x$.

These arrows define a natural isomorphism

$\eta: I \circ F \Rightarrow id_{G \ltimes X}$.

Naturality follows from

$g\, s_x = s_y\, F(g)$.

Indeed,

$\mathbf{s_y F(g)}$

$\mathbf{s_y s_y^{-1} g s_x}$

gs_x.

Thus F and I exhibit an equivalence

$G \ltimes X \simeq BH$.

7.6 Application to the Seed Orbit

Now take

$G=S_3$

and

$X=Orb(w)$.

Choose the original table w as base object.

Its stabilizer is

$H=Aut(A,w)=1$.

Therefore,

$S_3 \ltimes Orb(w) \simeq B1$.

The groupoid B1 is the terminal groupoid.

Hence

$S_3 \ltimes \mathrm{Orb}(w) \simeq *$.

The six labeled seed presentations form a contractible moduli groupoid.

This conclusion is stronger than saying that the orbit set has one element after quotienting.

It says:

all six presentations are connected by explicit isomorphisms,

every such comparison is unique,

and no presentation has nontrivial self-symmetry.

7.7 Full Faithfulness in the Rigid Case

Because the action is free and transitive, for every $u,v \in \mathrm{Orb}(w)$,

$\mathrm{Hom}(u,v)$

contains exactly one arrow.

The terminal groupoid has exactly one arrow from its unique object to itself.

Therefore the functor to the terminal groupoid is fully faithful:

$\mathrm{Hom}(u,v) \to \mathrm{Hom}(,)$

is a bijection for all u,v.

It is also essentially surjective because the terminal groupoid has one object.

Thus the equivalence can be verified without any abstract machinery.

The action groupoid is a six-object presentation of the terminal groupoid.

7.8 Nerves of Equivalent Groupoids

The nerve functor sends a groupoid to a simplicial set.

An n-simplex in the nerve of a groupoid is a composable string of n arrows.

An equivalence of groupoids induces a weak homotopy equivalence of their nerves.

Therefore,

$$N(S_3 \ltimes Orb(w))$$
$$\simeq$$
$$N(B1).$$

Since N(B1) is the constant one-point simplicial set,

$$|N(S_3 \ltimes Orb(w))|$$
$$\simeq$$
$$*.$$

Thus the classifying space of the seed action groupoid is contractible.

The many labeled presentations do not generate nontrivial homotopy because the action is free and transitive.

7.9 The Homotopy Quotient

For a group action $G \curvearrowright X$, the homotopy quotient is

$$X//G.$$

It may be modeled by the classifying space of the action groupoid:

$$X//G$$
$$:=$$
$$|N(G \ltimes X)|.$$

Equivalently,

$$X//G \simeq EG \times_G X.$$

For the seed orbit,

$$Orb(w)//S_3$$
$$\simeq$$
$$|N(S_3 \ltimes Orb(w))|.$$

Since Orb(w) is a free transitive S_3-set, it is isomorphic to S_3 with the regular left action.

Therefore,

$$ES_3 \times_{S_3} Orb(w)$$
$$\simeq$$
$$ES_3.$$

Since ES_3 is contractible,

$\mathrm{Orb}(w)//S_3 \simeq *$.

The ordinary quotient and homotopy quotient agree in homotopy type here because the action has no isotropy.

7.10 Ordinary Quotient Versus Homotopy Quotient

The ordinary quotient is

$\mathrm{Orb}(w)/S_3 = \{*\}$

because the action is transitive.

The homotopy quotient is also contractible:

$\mathrm{Orb}(w)//S_3 \simeq *$.

This agreement is not automatic for arbitrary actions.

For a one-point space with trivial G-action,

/G = {},

but

$*//G \simeq BG$.

The ordinary quotient forgets the stabilizer. The homotopy quotient retains it.

For the seed orbit, the stabilizer is trivial, so there is no hidden isotropy to retain.

The collapse to a point is therefore genuine rather than an artifact of discarding automorphisms.

7.11 Comparison with the Bare Carrier

Consider the bare three-element carrier A without the operation w.

Its full automorphism group is S_3.

The moduli groupoid of one unlabeled three-element set has skeleton

BS_3.

Its classifying space is

$B\,S_3$.

This space is not contractible.

Its fundamental group is

$\pi_1(BS_3) \cong S_3$.

For the structured seed,

Aut(A,w)=1,

so its moduli skeleton is

B1 ≃ *.

Thus adding the operation changes the moduli homotopy type from

BS_3

to

*.

The operation eliminates all isotropy.

This is a precise homotopical expression of rigidity.

7.12 Stabilizers as Residual Identity

In a moduli problem, automorphisms are loops.

For an object x, the loop group at x in the moduli groupoid is

Aut(x).

The stabilizer therefore measures the residual identity structure that remains after quotienting by presentation.

A nontrivial stabilizer produces nontrivial loops in the moduli space.

A trivial stabilizer produces no nonidentity loops.

For the seed,

Ω_x Moduli loops ≅ Aut(A,w)=1.

The seed has one isomorphism class and no residual internal symmetry.

Its moduli point is rigid.

7.13 Orbit–Stabilizer as Moduli Compression

The six labeled presentations contain redundant coordinate information.

The orbit–stabilizer equivalence compresses this redundancy:

six objects,

36 explicit relabeling arrows,

one isomorphism class,

one trivial stabilizer,

one terminal skeleton.

The compression preserves all categorical information.

It removes duplicated presentations but retains the automorphism group.

In the present case, nothing remains beyond one object and its identity.

Thus

$S_3 \ltimes \mathrm{Orb}(w)$

and

-

are not equal as categories, but they represent the same moduli problem up to equivalence.

7.14 The Classifying-Space Formula

For a transitive action $G \curvearrowright G/H$,

$G \ltimes G/H \simeq BH$.

Taking nerves and realizations gives

$(G/H)//G \simeq BH$.

This is the homotopy orbit–stabilizer formula.

For the seed,

$G=S_3$,

$H=\mathrm{Aut}(A,w)=1$.

Therefore,

$\mathrm{Orb}(w)//S_3$
$\simeq$
$B\,\mathrm{Aut}(A,w)$
$\simeq$

$$B1 \simeq *.$$

The final homotopy type is determined entirely by the seed automorphism group.

The six-element orbit does not contribute additional homotopy after the relabeling action is incorporated.

7.15 Limits of the Contractible Result

The contractibility of the seed moduli object is not a universal feature of three-element operations.

Other binary operations on A may have nontrivial stabilizers.

For such an operation u,

$$S_3 \ltimes Orb(u) \simeq B\,Aut(A,u),$$

and the resulting classifying space may have nontrivial fundamental group.

The present seed lies at the rigid extreme:

$$Aut(A,w)=1.$$

Thus the terminal endpoint depends on a specific computed property of w, not merely on the size of the carrier.

A larger moduli problem containing all binary operations on A would decompose into many connected components with different stabilizer groups.

The present construction isolates one rigid component.

7.16 Orbit–Stabilizer Theorem for the Seed

Theorem. Let S_3 act on the six relabeled presentations of the seed w. Then the action groupoid satisfies

$$S_3 \ltimes Orb(w) \simeq B\,Aut(A,w).$$

Since

Aut(A,w)=1,

it follows that

$S_3 \ltimes Orb(w)$
$\simeq$
$*$.

Consequently,

$Orb(w)//S_3$
$\simeq$
$*$,

and the nerve of the action groupoid is contractible.

The ordinary quotient contains one point, and the homotopy quotient contains no additional isotropy.

7.17 Transition to Univalent Moduli

The moduli branch now has the form

seed table

→ relabeling orbit

→ action groupoid

→ stabilizer group

→ classifying groupoid

→ homotopy quotient.

For the present seed,

six labeled presentations

→ one rigid moduli point.

The remaining step is to interpret this skeletal moduli object as a univalent category.

Because the skeleton has one object and one identity arrow, equality and isomorphism coincide exactly. Its simplicial nerve is Segal and complete.

This is the bounded univalent endpoint developed in the next section.

8. Rigidity, Rezk Completeness, and Bounded Univalence

8.1 The Skeletal Moduli Category

The seed action groupoid has six objects but only one isomorphism class. Its orbit–stabilizer reduction is

$S_3 \ltimes \mathrm{Orb}(w) \simeq B\,\mathrm{Aut}(A,w)$.

Since

$\mathrm{Aut}(A,w)=1$,

the reduced groupoid is

B1.

This is the terminal category, denoted 1.

It has:

one object *,

one morphism id*,

one isomorphism id*.

The six labeled seed tables have therefore been compressed to one skeletal moduli object without loss of categorical information.

The reduction removes duplicated presentations while preserving all automorphisms. Because the seed is rigid, no nonidentity automorphism remains.

8.2 Equality and Isomorphism

In an ordinary skeletal category, isomorphic objects are equal as objects of the chosen skeleton.

For the terminal category, there is only one object. Thus the object-equality relation is the singleton relation:

=.

The set of isomorphisms from * to * is also a singleton:

Iso(,)={id*}.

Therefore,

Id() ≃ *Iso(*,*).

Both sides contain exactly one element.

The map from equality to isomorphism sends reflexivity to the identity:

idtoiso(refl*)=id*.

Its inverse sends the unique isomorphism to reflexivity:

isotoid(id*)=refl*.

The composites are identities because both domains are singletons.

Thus equality and isomorphism coincide exactly at the skeletal moduli point.

8.3 The Meaning of Bounded Univalence

The univalence principle states, schematically, that equivalent structures may be identified.

At the categorical level, a univalent category is one in which equality of objects corresponds to isomorphism of objects.

The seed moduli category satisfies this principle:

$(x=y) \simeq Iso(x,y)$.

Because the category has only one object, the verification is finite and exact.

This is a bounded form of univalence. It concerns one finite 1-category obtained from one finite moduli problem.

It does not construct a universe containing arbitrary types or categories. It does not prove the general univalence axiom. It verifies the identity–isomorphism correspondence for the terminal skeletal completion of the seed orbit.

The correct statement is therefore:

the rigid seed determines a finite univalent moduli category.

8.4 The Nerve of the Terminal Category

Let N(1) be the simplicial nerve of the terminal category.

For every $n \geq 0$, an n-simplex is a string of n composable morphisms.

Since the only morphism is id*,

$N(1)_n=\{(,id,\ldots,id^*)\}$.

Thus every simplicial degree contains exactly one simplex.

Every face map is the unique map

$1 \to 1$.

Every degeneracy map is also the unique map

$1 \to 1$.

Hence N(1) is the constant one-point simplicial set.

Its geometric realization is a point:

$|N(1)| \cong *$.

The simplicial object contains no nontrivial loops, higher cells, or isotropy.

8.5 Segal Maps

For a simplicial space W, the Segal map in degree n compares n-fold composition data with strings of composable 1-simplices:

W_n
$\to$
$W_1 \times_{\{W_0\}} \cdots \times_{\{W_0\}} W_1$.

For the terminal nerve,

$W_0=1$,

$W_1=1$,

$W_n=1$

for every n.

The iterated fiber product of singleton spaces is a singleton.

Therefore every Segal map is

$1 \to 1$

and is a bijection, homeomorphism, and weak equivalence.

Thus the terminal nerve satisfies the Segal condition exactly.

Composition is not merely coherent up to homotopy. It is uniquely determined.

8.6 Equivalences in the Terminal Segal Space

Let W be the discrete simplicial space obtained from the terminal category.

The space W_1 consists of the unique arrow id*.

That arrow is an equivalence because it has itself as inverse.

Hence the subspace of equivalences is

W_1^eq = W_1 = 1.

The degeneracy map

s_0: W_0 → W_1^eq

sends the unique object to its identity arrow.

It is therefore the unique map

1 → 1

and is a bijection.

This is the completeness condition for the terminal Segal space.

Therefore W is a complete Segal space.

8.7 Rezk Completeness

A Segal space is Rezk-complete when the space of objects is equivalent to the space of equivalences through the identity map.

For the terminal moduli object,

$W_0 \simeq W_1$^eq $\simeq$ *.

The completeness map is an equivalence.

Thus the terminal skeleton is Rezk-complete.

The action groupoid itself contains six distinct objects connected by unique isomorphisms. Its ordinary discrete nerve is not skeletal at the object level. The passage to the terminal skeleton removes this presentational multiplicity.

The Rezk-complete object represents the moduli problem in a form where equivalent presentations are no longer treated as distinct objects.

8.8 Completion of the Action Groupoid

The original action groupoid is

$\mathcal{G} = S_3 \ltimes$ Orb(w).

It contains six labeled presentations.

Its complete skeletal representative is

$\mathcal{G}$ = 1.

The equivalence

$\mathcal{G} \simeq \mathcal{G}$

identifies all six presentations with the unique object *.

Every relabeling arrow is sent to id*.

Because there is exactly one arrow between every ordered pair in $\mathcal{G}$, this collapse remains fully faithful.

The completion does not arbitrarily erase distinctions. It removes only distinctions already witnessed as unique isomorphisms.

Thus the complete moduli object satisfies the principle:

presentations related by relabeling determine one object.

8.9 Rigidity and Uniqueness of Identity

The univalent endpoint depends critically on rigidity.

Suppose instead that the stabilizer were a nontrivial group H.

Then the skeletal moduli groupoid would be

BH.

It would have one object but several automorphisms.

In an ordinary set-level category, the equality type of the unique object would be a singleton, while

Iso(,) ≅ H

would contain multiple elements.

Therefore the ordinary one-object category BH would not satisfy set-level equality–isomorphism coincidence unless H were trivial.

For the present seed,

H=Aut(A,w)=1.

Hence the obstruction disappears.

Rigidity is therefore not a minor computational detail. It is the condition that permits the skeletal 1-category to be univalent without introducing higher identity structure.

8.10 Higher Identity and Nontrivial Stabilizers

A nontrivial automorphism group need not contradict univalence in homotopy type theory.

In a genuinely higher setting, the identity type of an object may itself contain nontrivial paths. Those paths can correspond to automorphisms.

A classifying type BH has one point but loop space

$\Omega BH \simeq H$.

Its identity structure is therefore richer than ordinary set equality.

The present package does not construct such higher identity types internally. Instead, it reaches univalence through the rigid case, where both identity and automorphism structure are trivial.

Thus the result lies at the 1-truncated boundary between ordinary categorical skeletonization and higher univalent moduli.

8.11 The Classifying-Space Perspective

The moduli groupoid has classifying space

$\mathbf{B}\mathcal{G}$

$|N(\mathcal{G})|$.

The orbit–stabilizer equivalence gives

$B\mathcal{G} \simeq B\,\mathrm{Aut}(A,w)$.

Since the automorphism group is trivial,

$B\mathcal{G} \simeq *$.

The loop space at the moduli point is therefore

$\Omega B\mathcal{G} \simeq \mathrm{Aut}(A,w) \simeq 1$.

This agrees with the categorical identity–isomorphism calculation.

The following three descriptions coincide:

the stabilizer is trivial,

the moduli classifying space has trivial loop space,

the skeletal moduli category has only the identity isomorphism.

Rigidity, contractibility, and bounded univalence are three views of the same terminal structure.

8.12 Presentation Identity Versus Structural Identity

At the table level, the six relabeled operations are unequal as literal arrays.

For $\sigma \neq \tau$,

$w^{\sigma} \neq w^{\tau}$

as labeled functions.

At the moduli level, they represent the same object because there is an isomorphism between them.

The action groupoid records this distinction:

literal equality separates the six tables,

isomorphism connects them.

The complete moduli category then identifies the isomorphic presentations.

Thus the development passes through three notions:

table equality,

algebra isomorphism,

moduli identity.

Univalence concerns the final relation between structural equivalence and identity, not literal equality of coordinate presentations.

8.13 A Finite Identity–Isomorphism Certificate

The univalence certificate can be reduced to finite counts.

For the skeletal category 1:

number of objects = 1,

number of object equalities = 1,

number of isomorphisms = 1,

number of equivalences = 1.

The maps

Id(,) → Iso(,)

and

Iso(,) → Id(,)

are inverse bijections.

For the corresponding complete Segal space:

$|W_o|=1$,

$|W_1\text{^}eq|=1$,

$s_{oo} \to W_1\text{^}eq$

is bijective.

Every Segal map is also a bijection.

The endpoint is therefore completely decidable by finite enumeration.

8.14 What Has Been Completed

The completion concerns the moduli of one seed under relabeling.

The input is not an arbitrary category. It is the finite action groupoid generated by the S_3-orbit of w.

The output is not a general univalent universe. It is the complete skeletal representative of that one connected rigid groupoid.

The completed structure retains:

one isomorphism class,

the automorphism group of the seed,

the homotopy type of the action groupoid.

For this seed, these data reduce to:

one object,

trivial automorphism group,

contractible homotopy type.

The completed object is therefore terminal.

8.15 What Has Not Been Constructed

The development does not provide:

a dependent type theory,

a universe closed under type formers,

a universe of finite algebras,

a general Rezk-completion algorithm,

higher inductive types,

computational path induction,

a model of cubical type theory,

a proof of universe-level univalence,

or nontrivial higher identity spaces.

The term univalent must therefore remain qualified.

The achieved result is:

finite,

categorical,

1-truncated,

seed-specific,

and exact.

Its strength lies in the complete derivational path from the original operation to the terminal moduli object.

8.16 The Two Homotopy Outcomes

The construction produces two different homotopy types in two different branches.

The proper-object boundary gives

$|N(\partial\Omega)| \simeq S^1$.

The seed moduli groupoid gives

$|N(S_3 \ltimes Orb(w))| \simeq *$.

These outcomes do not conflict.

The circle arises from incidence among selected objects of the free meet world.

The point arises from quotienting labeled seed presentations by relabeling while retaining stabilizers.

The first branch studies topology internal to a constructed world.

The second studies the moduli topology of presentations of the seed.

Thus:

world boundary → nontrivial loop,

seed moduli → rigid point.

The circle does not produce the univalent point. Both descend from the same finite seed through different construction policies.

8.17 Univalent Moduli Theorem

Theorem. Let

$\mathcal{G} = S_3 \ltimes Orb(w)$

be the action groupoid of relabeled presentations of the seed. Then

$\mathcal{G} \simeq B\,Aut(A,w)$.

Since

Aut(A,w)=1,

the groupoid $\mathcal{G}$ is equivalent to the terminal category 1.

The discrete simplicial nerve of 1 satisfies:

all Segal maps are bijections,

the equivalence subspace is a singleton,

the completeness map

$s_{oo} \to W_1^{eq}$

is a bijection.

Therefore the skeletal seed moduli category is Rezk-complete.

Moreover,

Id(*) ≃ *Iso*(*,*) ≃ 1.

The rigid seed thus admits a finite, exact, 1-truncated univalent moduli completion.

8.18 The Highest Justified Endpoint

The ascent can now be written as

one finite operation

→ full clone

→ derived meet

→ seven-object world

→ categorical nerve and circle boundary

and independently,

one finite operation

→ relabeling orbit

→ action groupoid

→ orbit–stabilizer equivalence

→ rigid skeletal moduli point

→ Rezk-complete univalent category.

The highest justified statement is not that one binary table generates all of homotopy type theory.

It is:

one binary table supports an explicit passage from finite universal algebra to a rigid univalent moduli 1-category.

This endpoint is mathematically narrow but structurally exact. It reaches univalence where equality and isomorphism can be certified completely, without concealing the selections required along the way.

9. Scope, Selection Policies, and Mathematical Limits

9.1 Derived Structure and Selected Structure

The construction contains both deduction and design.

The seed determines its term clone. The explicit terms r and m are derived inside that clone. Their algebraic properties are proved by finite evaluation.

By contrast, the following steps are selected:

$\Omega := F^{+}\Lambda(A)$,

removal of the cone point A,

choice of an orientation of the six-cycle,

choice of coefficient modules,

choice of monodromy automorphisms,

and passage to a skeletal moduli representative.

These selections are mathematically natural, but they are not uniquely forced by functional completeness.

The valid developmental relation is therefore not

seed logically necessitates every later object.

It is

seed supports a sequence of explicit, theory-preserving constructions.

9.2 The Four Logical Statuses

Each statement in the development belongs to one of four classes.

A computed statement is established by exhaustive finite enumeration.

Examples include

$|Clo_1(w)|=27$,

$|Clo_2(w)|=19{,}683$,

$Aut(A,w)=1$.

A derived statement is witnessed by explicit terms or maps.

Examples include

$r^2=r$,

$r(A)=B$,

$(B,m)\cong 2$.

A selected statement introduces a declared construction policy.

Examples include

$\Omega:=F^+\Lambda(A)$,

$\partial\Omega:=\Omega-\{A\}$.

An invoked statement applies a general theorem to certified finite data.

Examples include

$|N(\partial\Omega)|\cong S^1$,

$\Pi_1(S^1)\simeq B\mathbb{Z}$,

$G\ltimes G/H\simeq BH$.

This classification prevents selected constructions from being misreported as inevitable consequences.

9.3 Why the Free World Is Natural

The choice

$\Omega:=F^+\Lambda(A)$

is motivated by four properties.

First, it uses the derived meet algebra rather than an unrelated external structure.

Second, it reuses the original carrier A as the generator index set.

Third, it produces a finite object-world with exact universal properties.

Fourth, it admits complete reconstruction by three binary coordinate views.

The world is therefore canonical relative to the chosen category of nonempty meet-semilattices.

It is not canonical among all mathematical constructions available inside a functionally complete clone.

9.4 Why the Cone Point Is Removed

The full nerve $N(\Omega)$ is contractible because Ω has an initial object.

Removing A produces the proper Boolean boundary.

This deletion is selected because it separates interior completion from boundary incidence.

The resulting relation is

full free world → cone,

proper-object world → sphere boundary.

For three generators, the sphere boundary is S^1.

The nontrivial topology is therefore not hidden inside the full nerve as its homotopy type. It appears after removal of the universal lower object.

9.5 Independence of the Two Branches

The topological branch and the moduli branch share the same seed but answer different questions.

The topological branch asks:

What homotopy structure appears in the incidence geometry of the selected free world?

The moduli branch asks:

What remains after all labeled presentations of the seed are identified by relabeling?

The first yields

S^1.

The second yields

*.

There is no implication

S^1 → univalent point.

The two branches meet only at their common finite origin and through the natural relabeling action.

9.6 The Sphere Extension

The proper Boolean lattice on n+1 generators has order complex equivalent to S^{n-1}.

The present construction realizes the n=2 case.

Further spheres can be obtained by increasing the number of generators or by iterated suspension.

Such extensions are mathematically standard once a sphere has appeared.

They are not additional properties uniquely encoded by the original three-state operation.

The seed provides one explicit entrance to a sphere-generating architecture. It does not uniquely select an infinite tower of spheres.

9.7 Limits of Computational Certification

All finite claims can be exhaustively checked.

The operation table, clone closures, derived terms, group action, stabilizer, action groupoid, chain matrices, covering maps, and finite examples of local systems admit direct computation.

General mathematical conclusions still depend on classical theory.

Examples include:

the functional-completeness criterion,

contractibility of a category with an initial object,

the identification of Boolean boundaries with spheres,

classification of circle covers by subgroups of $\mathbb{Z}$,

equivalence of transitive action groupoids with stabilizer groupoids,

and the relation between groupoid equivalence and weak equivalence of nerves.

The computation certifies that the hypotheses hold. It does not replace every general theorem with a first-principles machine proof.

9.8 Python Verification and Formal Proof

The release uses deterministic finite computation.

This provides:

exact enumeration,

explicit witnesses,

repeatable reports,

independent reconstruction,

and exhaustive testing over finite domains.

It does not provide kernel-checked formalization in a proof assistant.

A proof-assistant development would require formal definitions of:

the seed algebra,

clone generation,

finite functional completeness,

free semilattices,

simplicial nerves,

covering spaces,

action groupoids,

and complete Segal spaces.

The present release is computationally certified mathematics, not a complete formalization in dependent type theory.

9.9 Minimality

The package begins with one binary operation on three elements.

This is minimal in the number of primitive operation symbols.

It is not proved globally minimal in carrier size, table complexity, term depth, or information content.

The two-element case cannot use the stated three-valued functional-completeness criterion in the same form. Other three-element operations may generate full clones or admit shorter bridges.

The valid claim is therefore:

the construction begins from one compact explicit seed.

The stronger claim

this is the unique or absolutely smallest possible seed

is not established.

9.10 Functional Completeness Is Not Explanation

A full clone contains every finitary operation on its carrier.

Consequently, it contains many incompatible internal structures.

One may derive groups, orders, selectors, constants, permutations, and arbitrary tables.

Functional completeness therefore supplies possibility, not uniqueness.

The explanatory content lies in the explicit developmental path:

which terms are selected,

which laws they satisfy,

which universal construction is applied,

which invariants are retained,

and which choices remain visible.

Without that discipline, every later structure could be dismissed as mere encoding.

9.11 Rigidity Limits the Final Moduli Complexity

The seed has trivial automorphism group.

This makes the final moduli category univalent in the simplest possible way.

It also makes the endpoint homotopically trivial.

A seed with nontrivial automorphisms would yield

B Aut(A,u)

rather than a point.

Such a moduli object would contain genuine loop structure.

To obtain a richer univalent moduli theory, one must either:

study seeds with nontrivial stabilizers,

consider many isomorphism classes simultaneously,

or move from set-level skeletal categories to higher identity types.

The present seed reaches the univalent boundary through rigidity rather than through nontrivial higher paths.

9.12 Limits of the Univalence Claim

The construction does not derive the univalence axiom for a universe.

It does not construct a universe closed under dependent products, dependent sums, identity types, or higher inductive types.

It does not formalize arbitrary equivalences as paths.

It does not establish nontrivial higher homotopy groups of a universe.

The achieved endpoint is narrower:

a finite action groupoid,

its terminal skeletal representative,

and exact coincidence of equality and isomorphism at that one rigid moduli point.

The phrase univalent moduli completion is valid only with these restrictions understood.

9.13 The Strongest Defensible Claim

The strongest compact statement supported by the construction is:

There exists an explicit three-element binary operation whose term clone is full, whose term language contains a two-element meet retract, whose selected free world has a proper boundary realizing S^1, and whose relabeling orbit forms a rigid action groupoid equivalent to a terminal Rezk-complete category.

This statement joins all major results without collapsing their logical distinctions.

9.14 Theory-Preserving Ontogenesis

Theory-preserving ontogenesis requires that each transition preserve an account of its origin.

The seed table is recorded.

Term witnesses are explicit.

Construction policies are declared.

Naturality under relabeling is checked.

Topological identifications are supported by finite incidence data.

Moduli equivalences retain stabilizer information.

Completion removes only distinctions already identified by isomorphism.

The result is not emergence without assumptions.

It is controlled mathematical development without concealed assumptions.

9.15 Frontier

The immediate frontier is the full moduli groupoid of all binary operations on a three-element carrier.

There are

$3^9 = 19{,}683$

such operations.

The S_3-action partitions them into isomorphism classes with stabilizers drawn from the subgroup structure of S_3.

This larger moduli object would contain:

many connected components,

rigid and nonrigid seeds,

different automorphism groups,

different clone types,

and different derived algebraic worlds.

The present seed would appear as one rigid component.

Such a classification would move the project beyond terminal univalence toward a finite but genuinely heterogeneous moduli universe.

10. Conclusion

The development begins with one binary operation

$w: A \times A \to A$

on the three-element carrier

$A = \{0,1,2\}$.

No order, topology, category, group, or identity structure is assumed.

The seed generates the full positive-arity clone:

$Clo(w)=O_a$.

Explicit terms then produce an idempotent retraction

$r: A \to B$

with

$B=\{0,2\}$,

and a binary operation m making B a two-element meet-semilattice.

A declared free-construction policy forms

$\Omega:=F^{+}\Lambda(A)$.

This world has seven objects and admits the equivalent descriptions

$\Omega \cong P(A)-\{\emptyset\}$

and

$\Omega \cong B^{A}-\{\top^{A}\}$.

Its objects are reconstructed exactly by three binary coordinate views.

The earlier development ended at this seven-object self-indexing world.

The present development categorifies Ω.

As a thin category, Ω has an initial object and therefore a contractible nerve. Removing the full-support object exposes the proper Boolean boundary

$\partial\Omega=P(A)-\{\emptyset,A\}$.

Its nerve is the barycentric subdivision of the boundary of a triangle:

$N(\partial\Omega) \cong Sd(\partial\Delta^{2})$.

Hence

$|N(\partial\Omega)| \cong S^{1}$.

The finite incidence world therefore contains a certified circle boundary.

The circle yields an explicit integer-line universal cover. Closed paths are classified by winding number, giving

$\pi_1(S^{1}) \cong \mathbb{Z}$

and

$\mathrm{Deck}(\mathbb{R}/S^1) \cong \mathbb{Z}$.

The fundamental groupoid reduces to

$\Pi_1(S^1) \simeq B\mathbb{Z}$.

Finite-index subgroups $q\mathbb{Z}$ classify connected cyclic covers. Actions of $\mathbb{Z}$ on modules classify local systems. For an integral local system determined by an automorphism T of a free abelian group M,

$H^0(S^1;M_T)=\ker(T-I)$,

$H^1(S^1;M_T)=\mathrm{coker}(T-I)$.

Thus the seed-supported circle extends naturally into covering theory, monodromy, cohomology, and representation theory.

A second branch begins directly from the relabeling symmetry of the seed.

The symmetric group S_3 acts on the six labeled presentations of w. The stabilizer is

$\mathrm{Aut}(A,w)=1$.

Therefore the action is free and transitive.

The resulting action groupoid

$S_3 \ltimes \mathrm{Orb}(w)$

has six objects and 36 arrows, with exactly one arrow between every ordered pair.

The groupoid form of orbit–stabilizer gives

$S_3 \ltimes \mathrm{Orb}(w) \simeq B\,\mathrm{Aut}(A,w)$.

Since the seed is rigid,

$S_3 \ltimes \mathrm{Orb}(w) \simeq B1 \simeq *$.

The six labeled tables therefore represent one moduli object with no nontrivial isotropy.

Passing to the skeletal representative gives the terminal category.

Its unique object has one equality and one isomorphism:

$\mathrm{Id}() \simeq Iso(,*) \simeq 1$.

Its simplicial nerve satisfies every Segal condition, and its completeness map

$W_0 \to W_1\text{^eq}$

is a bijection.

The resulting moduli category is therefore Rezk-complete and univalent in a finite, rigid, 1-truncated sense.

The two branches have different endpoints:

proper-world incidence yields S^1,

seed-presentation moduli yields *.

The circle records nontrivial topology internal to the selected free world.

The point records the absence of residual symmetry in the seed's moduli class.

Neither branch should be confused with the other.

The complete ascent is

one operation

→ full clone

→ derived meet

→ seven-object free world

→ circle boundary

→ universal cover and representations,

and independently,

one operation

→ relabeling orbit

→ action groupoid

→ stabilizer reduction

→ terminal Rezk-complete moduli category.

Every transition has a declared logical status.

The clone is computed and proved.

The bridge terms are explicit.

The free world and boundary deletion are selected.

The topological and categorical conclusions invoke standard theorems whose finite hypotheses are certified.

The final result is not a derivation of full homotopy type theory from one finite table.

It is a reproducible construction showing that a single compact operation can support a passage from finite universal algebra to topology, groupoids, homotopy quotients, and bounded univalence.

The result may be compressed to

$w \vdash \mathrm{Clo}(w)=O_a$,

$w \vdash r,m$,

$\Omega:=F^{+}\Lambda(A)$,

$|N(\Omega-\{A\})|\cong S^{1}$,

$S_3 \ltimes \mathrm{Orb}(w)\simeq B\,\mathrm{Aut}(A,w)$,

$\mathrm{Aut}(A,w)=1$,

and therefore

$\mathrm{Id}(\mathit{)}\simeq \mathit{Iso(},*)\simeq 1$.

One operation generates a complete language.

A derived meet organizes a finite world.

Its boundary carries a circle.

Its presentations collapse to one rigid univalent moduli point.

11. References

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